4 Vector Spaces

INTRODUCTORY EXAMPLE

Space Flight and Control Systems

Twelve stories high and weighing 75 tons, *Columbia* rose majestically off the launching pad on a cool Palm Sunday morning in April 1981. A product of ten years' intensive research and development, the first U.S. space shuttle was a triumph of control systems engineering design, involving many branches of engineering—aeronautical, chemical, electrical, hydraulic, and mechanical.

The space shuttle's control systems are absolutely critical for flight. Because the shuttle is an unstable airframe, it requires constant computer monitoring during atmospheric flight. The flight control system sends a stream of commands to aerodynamic control surfaces and 44 small thruster jets. Figure 1 shows a typical closed-loop feedback system that controls the pitch of the shuttle during flight. (The pitch is the elevation angle of the nose cone.) The junction symbols ($\otimes$) show where signals from various sensors are added to the computer signals flowing along the top of the figure.

Mathematically, the input and output signals to an engineering system are functions. It is important in applications that these functions can be added, as in Fig. 1, and multiplied by scalars. These two operations on functions have algebraic properties that are completely analogous to the operations of adding vectors in $\mathbb{R}^n$ and multiplying a vector by a scalar, as we shall see in Sections 4.1 and 4.8. For this reason, the set of all possible inputs (functions) is called a *vector space*. The mathematical foundation for systems engineering rests

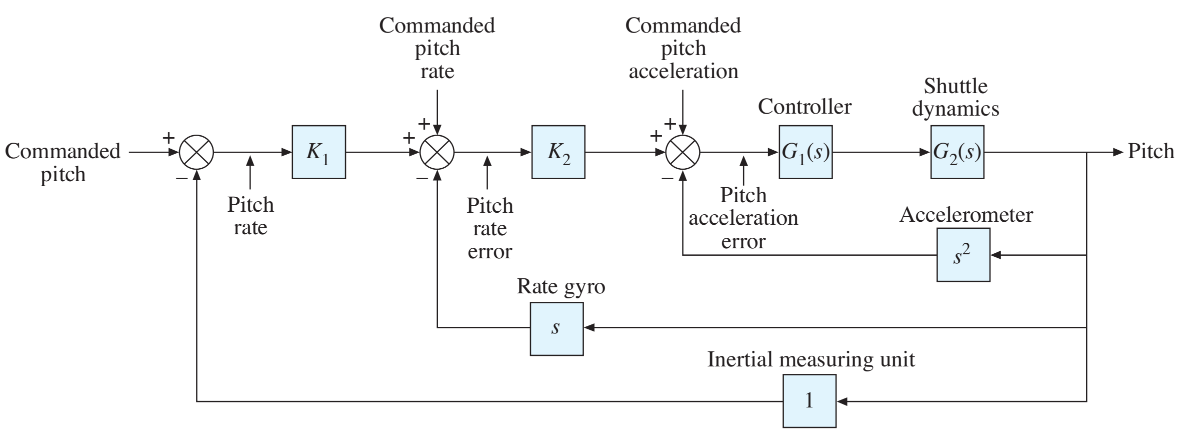

FIGURE 1 Pitch control system for the space shuttle. (*Source:* Adapted from *Space Shuttle GN&C Operations Manual*, Rockwell International, ©1988.)

on vector spaces of functions, and Chapter 4 extends the theory of vectors in $\mathbb{R}^n$ to include such functions. Later on, you will see how other vector spaces arise in engineering, physics, and statistics.

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The mathematical seeds planted in Chapters 1 and 2 germinate and begin to blossom in this chapter. The beauty and power of linear algebra will be seen more clearly when you view $\mathbb{R}^n$ as only one of a variety of vector spaces that arise naturally in applied problems. Actually, a study of vector spaces is not much different from a study of $\mathbb{R}^n$ itself, because you can use your geometric experience with $\mathbb{R}^2$ and $\mathbb{R}^3$ to visualize many general concepts.

Beginning with basic definitions in Section 4.1, the general vector space framework develops gradually throughout the chapter. A goal of Sections 4.3–4.5 is to demonstrate how closely other vector spaces resemble $\mathbb{R}^n$. Section 4.6 on rank is one of the high points of the chapter, using vector space terminology to tie together important facts about rectangular matrices. Section 4.8 applies the theory of the chapter to discrete signals and difference equations used in digital control systems such as in the space shuttle. Markov chains, in Section 4.9, provide a change of pace from the more theoretical sections of the chapter and make good examples for concepts to be introduced in Chapter 5.

4.1 VECTOR SPACES AND SUBSPACES

Much of the theory in Chapters 1 and 2 rested on certain simple and obvious algebraic properties of $\mathbb{R}^n$, listed in Section 1.3. In fact, many other mathematical systems have the same properties. The specific properties of interest are listed in the following definition.

DEFINITION

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

[1] Technically, V is a *real vector space*. All of the theory in this chapter also holds for a *complex vector space* in which the scalars are complex numbers. We will look at this briefly in Chapter 5. Until then, all scalars are assumed to be real.

Using only these axioms, one can show that the zero vector in Axiom 4 is unique, and the vector $-\mathbf{u}$, called the **negative** of $\mathbf{u}$, in Axiom 5 is unique for each $\mathbf{u}$ in V. See Exercises 25 and 26. Proofs of the following simple facts are also outlined in the exercises:

For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u} = \mathbf{0} \tag{1}$$

$$c\mathbf{0} = \mathbf{0} \tag{2}$$

$$-\mathbf{u} = (-1)\mathbf{u} \tag{3}$$

EXAMPLE 1 The spaces $\mathbb{R}^n$, where $n \geq 1$, are the premier examples of vector spaces. The geometric intuition developed for $\mathbb{R}^3$ will help you understand and visualize many concepts throughout the chapter. ■

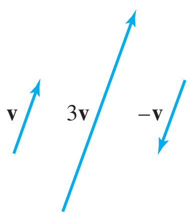

FIGURE 1

EXAMPLE 2 Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule (from Section 1.3), and for each $\mathbf{v}$ in V, define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of $\mathbf{v}$, pointing in the same direction as $\mathbf{v}$ if $c \geq 0$ and otherwise pointing in the opposite direction. (See Fig. 1.) Show that V is a vector space. This space is a common model in physical problems for various forces.

SOLUTION The definition of V is geometric, using concepts of length and direction. No xyz-coordinate system is involved. An arrow of zero length is a single point and represents the zero vector. The negative of $\mathbf{v}$ is $(-1)\mathbf{v}$. So Axioms 1, 4, 5, 6, and 10 are evident. The rest are verified by geometry. For instance, see Figs. 2 and 3. ■

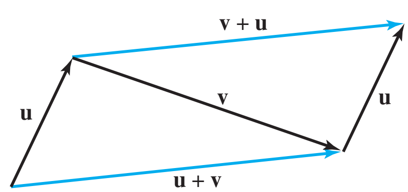

FIGURE 2 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

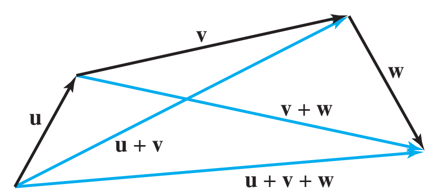

FIGURE 3 $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

EXAMPLE 3 Let $\mathbb{S}$ be the space of all doubly infinite sequences of numbers (usually written in a row rather than a column):

$$\{y_k\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$$

If $\{z_k\}$ is another element of $\mathbb{S}$, then the sum $\{y_k\} + \{z_k\}$ is the sequence $\{y_k + z_k\}$ formed by adding corresponding terms of $\{y_k\}$ and $\{z_k\}$. The scalar multiple $c\{y_k\}$ is the sequence $\{cy_k\}$. The vector space axioms are verified in the same way as for $\mathbb{R}^n$.

Elements of $\mathbb{S}$ arise in engineering, for example, whenever a signal is measured (or sampled) at discrete times. A signal might be electrical, mechanical, optical, and so on. The major control systems for the space shuttle, mentioned in the chapter introduction, use discrete (or digital) signals. For convenience, we will call $\mathbb{S}$ the space of (discrete-time) **signals**. A signal may be visualized by a graph as in Fig. 4. ■

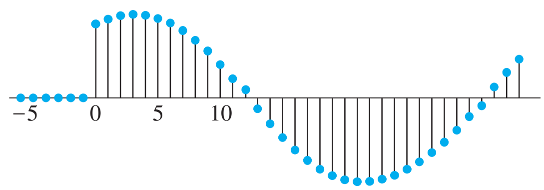

FIGURE 4 A discrete-time signal.

EXAMPLE 4 For $n \geq 0$, the set $\mathbb{P}_n$ of polynomials of degree at most n consists of all polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \tag{4}$$

where the coefficients $a_0, \ldots, a_n$ and the variable t are real numbers. The *degree* of $\mathbf{p}$ is the highest power of t in (4) whose coefficient is not zero. If $\mathbf{p}(t) = a_0 \neq 0$, the degree of $\mathbf{p}$ is zero. If all the coefficients are zero, $\mathbf{p}$ is called the *zero polynomial*. The zero polynomial is included in $\mathbb{P}_n$ even though its degree, for technical reasons, is not defined.

If $\mathbf{p}$ is given by (4) and if $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$, then the sum $\mathbf{p} + \mathbf{q}$ is defined by

$$\begin{aligned}(\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n\end{aligned}$$

The scalar multiple $c\mathbf{p}$ is the polynomial defined by

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = ca_0 + (ca_1)t + \cdots + (ca_n)t^n$$

These definitions satisfy Axioms 1 and 6 because $\mathbf{p} + \mathbf{q}$ and $c\mathbf{p}$ are polynomials of degree less than or equal to n. Axioms 2, 3, and 7–10 follow from properties of the real numbers. Clearly, the zero polynomial acts as the zero vector in Axiom 4. Finally, $(-1)\mathbf{p}$ acts as the negative of $\mathbf{p}$, so Axiom 5 is satisfied. Thus $\mathbb{P}_n$ is a vector space.

The vector spaces $\mathbb{P}_n$ for various n are used, for instance, in statistical trend analysis of data, discussed in Section 6.8. ■

EXAMPLE 5 Let V be the set of all real-valued functions defined on a set $\mathbb{D}$. (Typically, $\mathbb{D}$ is the set of real numbers or some interval on the real line.) Functions are added in the usual way: $\mathbf{f} + \mathbf{g}$ is the function whose value at t in the domain $\mathbb{D}$ is $\mathbf{f}(t) + \mathbf{g}(t)$. Likewise, for a scalar c and an $\mathbf{f}$ in V, the scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$. For instance, if $\mathbb{D} = \mathbb{R}$, $\mathbf{f}(t) = 1 + \sin 2t$, and $\mathbf{g}(t) = 2 + .5t$, then

$$(\mathbf{f} + \mathbf{g})(t) = 3 + \sin 2t + .5t \quad \text{and} \quad (2\mathbf{g})(t) = 4 + t$$

Two functions in V are equal if and only if their values are equal for every t in $\mathbb{D}$. Hence the zero vector in V is the function that is identically zero, $\mathbf{f}(t) = 0$ for all t, and the negative of $\mathbf{f}$ is $(-1)\mathbf{f}$. Axioms 1 and 6 are obviously true, and the other axioms follow from properties of the real numbers, so V is a vector space. ■

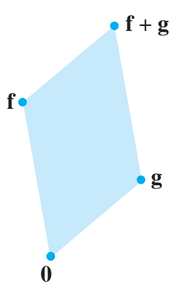

FIGURE 5 The sum of two vectors (functions).

It is important to think of each function in the vector space V of Example 5 as a single object, as just one "point" or vector in the vector space. The sum of two vectors $\mathbf{f}$ and $\mathbf{g}$ (functions in V, or elements of *any* vector space) can be visualized as in Fig. 5, because this can help you carry over to a general vector space the geometric intuition you have developed while working with the vector space $\mathbb{R}^n$. See the *Study Guide* for help as you learn to adopt this more general point of view.

Subspaces

In many problems, a vector space consists of an appropriate subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

DEFINITION

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

Properties (a), (b), and (c) guarantee that a subspace H of V is itself a *vector space*, under the vector space operations already defined in V. To verify this, note that properties (a), (b), and (c) are Axioms 1, 4, and 6. Axioms 2, 3, and 7–10 are automatically true in H because they apply to all elements of V, including those in H. Axiom 5 is also true in H, because if $\mathbf{u}$ is in H, then $(-1)\mathbf{u}$ is in H by property (c), and we know from equation (3) on page 191 that $(-1)\mathbf{u}$ is the vector $-\mathbf{u}$ in Axiom 5.

So every subspace is a vector space. Conversely, every vector space is a subspace (of itself and possibly of other larger spaces). The term *subspace* is used when at least two vector spaces are in mind, with one inside the other, and the phrase *subspace of V* identifies V as the larger space. (See Fig. 6.)

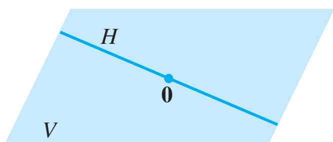

FIGURE 6
A subspace of V.

EXAMPLE 6 The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$. ■

EXAMPLE 7 Let $\mathbb{P}$ be the set of all polynomials with real coefficients, with operations in $\mathbb{P}$ defined as for functions. Then $\mathbb{P}$ is a subspace of the space of all real-valued functions defined on $\mathbb{R}$. Also, for each $n \geq 0$, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$, because $\mathbb{P}_n$ is a subset of $\mathbb{P}$ that contains the zero polynomial, the sum of two polynomials in $\mathbb{P}_n$ is also in $\mathbb{P}_n$, and a scalar multiple of a polynomial in $\mathbb{P}_n$ is also in $\mathbb{P}_n$. ■

EXAMPLE 8 The vector space $\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. (The vectors in $\mathbb{R}^3$ all have three entries, whereas the vectors in $\mathbb{R}^2$ have only two.) The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

is a subset of $\mathbb{R}^3$ that "looks" and "acts" like $\mathbb{R}^2$, although it is logically distinct from $\mathbb{R}^2$. See Fig. 7. Show that H is a subspace of $\mathbb{R}^3$.

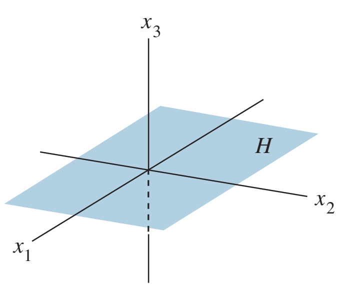

FIGURE 7
The x_1x_2-plane as a subspace of $\mathbb{R}^3$.

SOLUTION The zero vector is in H, and H is closed under vector addition and scalar multiplication because these operations on vectors in H always produce vectors whose third entries are zero (and so belong to H). Thus H is a subspace of $\mathbb{R}^3$. ■

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[2] Some texts replace property (a) in this definition by the assumption that H is nonempty. Then (a) could be deduced from (c) and the fact that $0\mathbf{u} = \mathbf{0}$. But the best way to test for a subspace is to look first for the zero vector. If $\mathbf{0}$ is in H, then properties (b) and (c) must be checked. If $\mathbf{0}$ is *not* in H, then H cannot be a subspace and the other properties need not be checked.

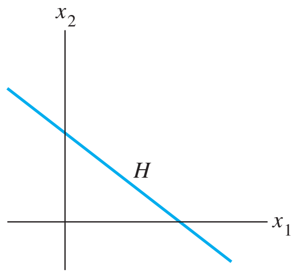

FIGURE 8
A line that is not a vector space.

EXAMPLE 9 A plane in $\mathbb{R}^3$ *not* through the origin is not a subspace of $\mathbb{R}^3$, because the plane does not contain the zero vector of $\mathbb{R}^3$. Similarly, a line in $\mathbb{R}^2$ *not* through the origin, such as in Fig. 8, is *not* a subspace of $\mathbb{R}^2$. ■

A Subspace Spanned by a Set

The next example illustrates one of the most common ways of describing a subspace. As in Chapter 1, the term **linear combination** refers to any sum of scalar multiples of vectors, and Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$.

EXAMPLE 10 Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

SOLUTION The zero vector is in H, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$. To show that H is closed under vector addition, take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

By Axioms 2, 3, and 8 for the vector space V,

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$

So $\mathbf{u} + \mathbf{w}$ is in H. Furthermore, if c is any scalar, then by Axioms 7 and 9,

$$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$

which shows that $c\mathbf{u}$ is in H and H is closed under scalar multiplication. Thus H is a subspace of V. ■

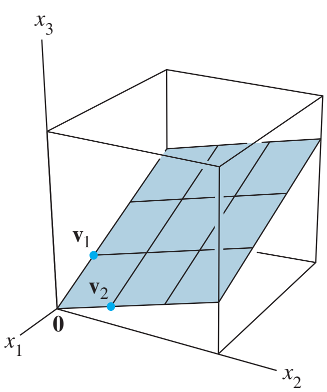

FIGURE 9
An example of a subspace.

In Section 4.5, you will see that every nonzero subspace of $\mathbb{R}^3$, other than $\mathbb{R}^3$ itself, is either Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for some linearly independent $\mathbf{v}_1$ and $\mathbf{v}_2$ or Span $\{\mathbf{v}\}$ for $\mathbf{v} \neq \mathbf{0}$. In the first case, the subspace is a plane through the origin; in the second case, it is a line through the origin. (See Fig. 9.) It is helpful to keep these geometric pictures in mind, even for an abstract vector space.

The argument in Example 10 can easily be generalized to prove the following theorem.

THEOREM 1

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V.

We call Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning** (or **generating**) **set** for H is a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

The next example shows how to use Theorem 1.

EXAMPLE 11 Let H be the set of all vectors of the form $(a - 3b, b - a, a, b)$, where a and b are arbitrary scalars. That is, let $H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{ in } \mathbb{R}\}$. Show that H is a subspace of $\mathbb{R}^4$.

SOLUTION Write the vectors in H as column vectors. Then an arbitrary vector in H has the form

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\uparrow\ \mathbf{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\uparrow\ \mathbf{v}_2}$$

This calculation shows that $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors indicated above. Thus H is a subspace of $\mathbb{R}^4$ by Theorem 1. ■

Example 11 illustrates a useful technique of expressing a subspace H as the set of linear combinations of some small collection of vectors. If $H = \text{Span}\,\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, we can think of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ in the spanning set as "handles" that allow us to hold on to the subspace H. Calculations with the infinitely many vectors in H are often reduced to operations with the finite number of vectors in the spanning set.

EXAMPLE 12 For what value(s) of h will $\mathbf{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

SOLUTION This question is Practice Problem 2 in Section 1.3, written here with the term *subspace* rather than $\text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The solution there shows that $\mathbf{y}$ is in $\text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$. That solution is worth reviewing now, along with Exercises 11–16 and 19–21 in Section 1.3. ■

Although many vector spaces in this chapter will be subspaces of $\mathbb{R}^n$, it is important to keep in mind that the abstract theory applies to other vector spaces as well. Vector spaces of functions arise in many applications, and they will receive more attention later.

PRACTICE PROBLEMS

1. Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector $\mathbf{u}$ in H and a scalar c such that $c\mathbf{u}$ is not in H.)
2. Let $W = \text{Span}\,\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, where $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V. Show that $\mathbf{v}_k$ is in W for $1 \le k \le p$. [*Hint:* First write an equation that shows that $\mathbf{v}_1$ is in W. Then adjust your notation for the general case.]

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4.1 EXERCISES

1. Let V be the first quadrant in the xy-plane; that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \ge 0, y \ge 0 \right\}$$

a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?

b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is *not* in V. (This is enough to show that V is *not* a vector space.)

2. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \ge 0 \right\}$.

a. If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

b. Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u}+\mathbf{v}$ is not in W. This is enough to show that W is *not* a vector space.

3. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1\right\}$. Find a specific example—two vectors or a vector and a scalar—to show that H is not a subspace of $\mathbb{R}^2$.

4. Construct a geometric figure that illustrates why a line in $\mathbb{R}^2$ *not* through the origin is not closed under vector addition.

In Exercises 5–8, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

5. All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

6. All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

7. All polynomials of degree at most 3, with integers as coefficients.

8. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.

9. Let H be the set of all vectors of the form $\begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\,\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

10. Let H be the set of all vectors of the form $\begin{bmatrix} 3t \\ 0 \\ -7t \end{bmatrix}$, where t is any real number. Show that H is a subspace of $\mathbb{R}^3$. (Use the method of Exercise 9.)

11. Let W be the set of all vectors of the form $\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

12. Let W be the set of all vectors of the form $\begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$. (Use the method of Exercise 11.)

13. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a. Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

c. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

14. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be as in Exercise 13, and let $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 14 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

In Exercises 15–18, let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

15. $\begin{bmatrix} 2a+3b \\ -1 \\ 2a-5b \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 3a-5b \\ 3b+2a \end{bmatrix}$

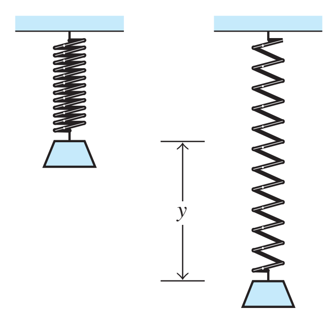
17. $\begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix}$

18. $\begin{bmatrix} 4a+3b \\ 0 \\ a+3b+c \\ 3b-2c \end{bmatrix}$

19. If a mass m is placed at the end of a spring, and if the mass is pulled downward and released, the mass–spring system will begin to oscillate. The displacement y of the mass from its resting position is given by a function of the form

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t \qquad (5)$$

where ω is a constant that depends on the spring and the mass. (See the figure below.) Show that the set of all functions described in (5) (with ω fixed and c_1, c_2 arbitrary) is a vector space.



20. The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

a. What facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed? (These facts are usually discussed in a calculus class.)

b. Show that $\{\mathbf{f} \text{ in } C[a, b] : \mathbf{f}(a) = \mathbf{f}(b)\}$ is a subspace of $C[a, b]$.

For fixed positive integers m and n, the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

21. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

22. Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA = 0$ (the zero matrix in $M_{3\times 4}$). Determine if H is a subspace of $M_{2\times 4}$.